Categorical Models for Relevant Logics

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Abstract

In this note we try to understand and present some of the relevant logics in what we think is the mathematically tidier way. Since we want to use Category Theory for our models, we want to have (cut-free) Gentzen-style sequent calculus formulations for our systems. We present the systems, discuss their semantics (categorical and algebraic) and then relate them to the existing Relevant Logics.

KEYWORDS: Gentzen sequent calculus, Linear Logic, Relevant Logics, Category Theory, Categorical Logic.

1 Introduction

Intuitionistic Propositional Logic when originally presented by Gentzen in a sequent-calculus formulation **LJ** has three structural rules Permutation, Contraction and Weakening:

$$(P)\frac{\Gamma, A, B, \Gamma' \vdash C}{\Gamma, B, A, \Gamma' \vdash C} \qquad (C)\frac{\Gamma, A, A, \Delta \vdash C}{\Gamma, A, \Delta \vdash C} \qquad (W)\frac{\Gamma, \Delta \vdash C}{\Gamma, A, \Delta \vdash C}$$

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There has been recently a reasonable amount of work on the hierarchy of logics that result when we get rid of some (or all) of the structural rules. The resulting logical systems have been called *substructural logics*. The lack of the rule of Contraction means that each occurrence of a formula counts as a different formula, and so the contexts on the left of the turnstile must correspond to multisets and not only to sets. This accounts for the name of *occurrence logics*, and they have the particularity that the complexity of a sequent must increase in the course of a proof. Also, the lack of the Weakening rule shows that the logic is *non-monotonic* in the weak sense that if we are able to derive a formula C from a multiset Γ of premisses we can say nothing about the same C being derivable from Γ plus other premisses. It also means that in the antecedent of the sequent we must only put assumptions that are important (relevant) to the derivation of a formula.

Girard's Linear Logic gets rid of the two structural rules of Contraction and Weakening in one go (neither they hold nor they are derivable rules). As a result we obtain the well-known splitting of the logical conjunction and disjunction into two versions: one multiplicative and one additive. Also, the (multiplicative) implication we get behaves 'like' a relevant implication.

In this note we try to understand and present some of the relevant logics in what we think is the mathematically tidier way. Since we want to use Category Theory for our models, we want to have (cut-free) Gentzen-style sequent calculus formulations for our systems. We present the systems, discuss their semantics (categorical and algebraic) and then relate them to the existing Relevant Logics.

2 Tensor-Implication Logic

As it is usual in relevantists' work, we will begin our development by presenting the system that deals with the fragment implicational of the logic and we will add the other connectives later. But, because we want to work with Category Theory we believe that it is easier to begin with the fragment which has the multiplicative conjunction, or tensor $\otimes$ (and its respective unit), and the implication $\multimap$ connectives. We present a Gentzen-style sequent calculus and call this system **TI** :

Structural Rules:

$$(Identity)\frac{}{A \vdash A} \qquad (P)\frac{\Gamma, A, B, \Gamma' \vdash C}{\Gamma, B, A, \Gamma' \vdash C} \qquad (Cut)\frac{\Gamma \vdash A \quad A, \Delta \vdash B}{\Gamma, \Delta \vdash B}$$

Logical Rules:

$$I_l\frac{\Gamma \vdash A}{\Gamma, I \vdash A} \qquad I_r\frac{}{\vdash I}$$

$$\multimap_l \frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C} \qquad \multimap_r \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}$$

$$\otimes_l\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C} \qquad \otimes_r\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}$$

Due to the lack of traditional rules of Contraction and Weakening, in this calculus capital Greek letters (eg. Γ) stand for (finite) multisets (or bags) of assumptions. It might help the reader to think of the multiplicative conjunction or tensor connective $\otimes$ as a *juxtaposition* of data, so one can reorder data but one cannot duplicate or erase it. Tensor is a weak form of conjunction and the constant I is the identity for it. The linear implication $A \multimap B$ can be thought of a function which looks at its argument exactly once.

This calculus admits cut elimination [1], i.e. every derivation which uses the rule (*Cut*) can be transformed into a cut free derivation. This has very pleasant consequences: the consistency of the system and the subformula property (a cut free proof contains only subformulas of the sequent it proves). The system **TI** above is called in some papers with computational linguistic motivations the Lambek Calculus with Permutation **LP** ([2]).

The lack of Weakening means that in the antecedent of the sequent we must only put assumptions that are important (relevant) to derive a formula. Hence, we think of **TI** as one of the simplest Relevant Logics[1]. The basic idea of Relevant Logic is to constraint odd entailments and so, a well formed implicational formula is one where the antecedent really produce the consequent.

The system **TI** corresponds to the implicational plus cotenability (and the constant called t) fragment of the relevant logic **RW** ([4],[5]).

There are several different notations for algebraic models of **TI**. We follow the algebraic approach of thinking first of tensor logic and then adding implication. Thus, to model tensor logic we use any *ordered monoid* [6], the idea is that the partial order $\leq$ models entailment and the monoidal operation $\circ$ models both the comma and the tensor connective. To model tensor-implication logic we use any *closed poset* [6].

Categorical Logic is a well-established branch of Mathematical Logic and we take it for granted in this work. As every poset is a trivial kind of category (the category where all the maps from A to B are the "same" map) it makes sense to

[1] as a relevant logic it is a paraconsistent logic too [3].

look for more general models which preserve the distinction of maps. This is an oversimplified description but one can say that the natural generalization of the concept of an ordered symmetric monoid is a *symmetric monoidal category* [7], and the generalization of a closed poset is a *symmetric monoidal closed category* [7].

3 Extensions of the Tensor-Implication Calculus

When we are working with a sequent-calculus with all the structural rules, like **LJ**, we have that these structural rules hold in the sense up-to-down and then, with the help of the other rules, it is possible to derive the other direction. But, when any of the structural rule lacks we are not more able to conclude all the senses down-to-up from the permitted rules. So, if we want only to drop a structural rule of the calculus but preserving the whole effect of the other rules, we must add whichever of the other senses of the rules is lacking. Thus, for example, if we drop the rule of Weakening it is not more possible to conclude that the converse of the Contraction rule is valid, and this is important when we are worried with the number of times a formula appears in a derivation.

Therefore, we have that the system **TI** plus Contraction$\downarrow$ is the same as the implicational fragment of Relevant Logic $\mathbb{R}$ with cotenability and the constant t ([4],[5]). And, on the other hand, if we think that the converse of the Contraction (called *premise repetition*) is also allowed we obtain another traditional relevance logic, $\mathbf{RM0}_{\rightarrow}$ ([4], [5]). The system $\mathbf{RM0}_{\rightarrow}$ has the implication axioms of the relevant logic plus the mingle axiom. Meyer has shown that **RM** (the system obtained adding the mingle axiom to the full system $\mathbb{R}$) is not a conservative extension of

$\mathbf{RM0}_{\rightarrow}$.

We have that the algebraic structure needed to model **TI** plus Contraction($\downarrow$) is any closed poset $(M, \leq, \circ, e, \multimap)$ where $\circ$ is *square-increasing*, i.e., $a \leq a \circ a$ for all a in M. We call this structure a *Church monoid.*

If the monoidal operation $\circ$ is also *square-decreasing*, i.e. $a \circ a \leq a$ then, $\circ$ is *idempotent* and we have a model of **TI** plus Contraction.

As in the previous section we want to generalise the ideas of the algebraic models. The categorical generalization of a Church monoid is is a *symmetric monoidal category with diagonals* [12]. And, on the other hand if we want to model the converse of the Contraction rule we need the 'converse' notion of having diagonals. This corresponds to the concept of *monoid* [7]. So, a symmetric monoidal closed category **C** with diagonals where each object is a monoid models **TI** plus Contraction.

If we add to the system **TI** the rules corresponding to the additives & and $\oplus$ (with their units 1 and 0 respectively) we obtain the system corresponding to the Intuitionistic Linear Logic **ILL**. The algebraic models are the closed posets with finite meets and joints and the categorical structures are the symmetric monoidal closed categories with finite products and coproducts (called *linear categories*).

If we add to **TI** plus Contraction an axiom corresponding to the distributivity of the additives & and $\oplus$, i.e.the scheme axiom $\Gamma, A \,\&\, (B \oplus C) \vdash \Gamma, (A \,\&\, B) \oplus C$, we obtain the corresponding system to the relevant logic **R** plus distributivity (with cotenability). We can model it easily with distributive structures.

Also, if we add to **ILL** the rules of Contraction and Weakening we obtain the Intuitionistic Logic system **LJ** ([13]) because with these two structural rules we can prove that the connectives $\otimes$ and & are the same. We can model it through *cartesian closed categories with coproducts* [14].

Finally, we want to describe a semantical way of adding the structural rules of Contraction to Intuitionistic Linear Logic to get a relevant logic. This idea has been independently described by Prof. Kosta Dosen [15] but in that paper semantics is not presented. By contrast here the (categorical) semantics is the main motivation because the model so obtained is very elegant but now we will have in the calculus the **S4**-like rules that are difficult to deal with.

The idea is simple enough. We start with a sequent presentation of **ILL** and we add a modality, which we call $\Box$. This $\Box$ modality satisfies the **S4**-like rules ([16]) as well as the Contraction rule. We obtain that: the sequents provable in the calculus correspond to the sequents provable in the relevant logic **R** (without distributivity of the additives); and we obtain easily categorical models for this calculus.

4 Conclusions

A lot of work needs to be done. First as we would like to understand the whole family of Relevant Logics systems like E and T need to be analyzed. Also we need to develop term assigment systems and λ-calculus for all the logics discused here and to study the models like Kripke.

Finally, we want to make connections with several areas of Computer Science more explicit, mainly with some formal methods to model concurrency.

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